On the seismic inverse problem: uniqueness, stability and reconstruction

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Collaborators

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Outline

1. Motivation: Reflection Seismology
2. Mathematical model:
   time harmonic case, linearized elasticity, isotropic medium
3. The inverse problem
4. Some history
5. Regularization: Unknown parameters are piecewise constant on a finite partition of the background domain
6. Parameter identification (given the partition)
7. Quantitative Lipschitz stability estimates
8. Reconstruction algorithm
9. Partition identification
10. Final remarks

Part I

Formulation of the problem

Motivation

- Reflection Seismology
- Nondestructive testing of materials

Model

Time harmonic elastic wave equation

$$\mathrm{div}(\mathbb{C}\widehat{\nabla} u) + \omega^2 \rho u = 0$$

where $\mathbb{C}$ is the elasticity isotropic tensor and ρ the density, ω frequency

Lamé system of elasticity ($\omega = 0$)

$$\mathrm{div}(\mathbb{C}\widehat{\nabla} u) = 0$$

Mathematical formulation

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla} u) + \omega^2 \rho u &= 0 \text{ in } \Omega \subset \mathbb{R}^3 \\ u &= \psi \text{ on } \partial\Omega, \end{cases}$$

$\widehat{\nabla} u := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ **strain tensor**

$\psi \in H^{1/2}(\partial\Omega)$ **boundary displacement field**,

$\mathbb{C}$ **elasticity tensor**: isotropic, bounded and strongly convex

$$\mathbb{C} = \lambda I_3 \otimes I_3 + 2\mu \mathbb{I}_{sym},$$

$$\alpha_0 \leq \mu \leq \alpha_0^{-1}, \quad \beta_0 \leq 2\mu + 3\lambda \leq \beta_0^{-1},$$

ρ **density**

$$\gamma_0 \leq \rho \leq \gamma_0^{-1}$$

Well-posedness of the direct problem

Let λ_1^0 be the smallest Dirichlet eigenvalue of the operator $-\operatorname{div}(\mathbb{C}_0\hat{\nabla})$ in Ω, where $\mathbb{C}_0 = \frac{\beta_0 - 3\alpha_0}{2} I_3 \otimes I_3 + 2\alpha_0 \mathbb{I}_{sym}$ (so that $\mathbb{C} \geq \mathbb{C}_0$). Then, for any

$$\omega^2 \in (0, \frac{\gamma_0 \lambda_1^0}{2}]$$

there exists a unique weak solution $u \in H^1(\Omega)$ of

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla} u) + \omega^2 \rho u &= 0 \text{ in } \Omega \subset \mathbb{R}^3 \\ u &= \psi \text{ on } \partial\Omega. \end{cases}$$

Define the Dirichlet to Neumann map $\Lambda_{\mathbb{C},\rho} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$

$$\Lambda_{\mathbb{C},\rho}\psi = (\mathbb{C}\widehat{\nabla} u)\nu|_{\partial\Omega}$$

Formulation of the inverse problem

Seismic inverse problem

Determine $\mathbb{C} = (\mu, \lambda)$ and ρ from knowledge of the Dirichlet-to Neumann map $\Lambda_{\mathbb{C},\rho}$

Key issues

- **Uniqueness**
- **Stability**
- **Reconstruction**

Uniqueness

Static elastic case: $\omega = 0$

- Ikehata (1990): **linearized version**.
- Akamatsu, Nakamura and Steinberg (1991) **2D** , Nakamura, Uhlmann (1995) **3D**
  $\lambda, \mu \in C^\infty$ **implies determination of** λ, μ **and their derivatives on the boundary of a smooth domain**.
- Nakamura and Uhlmann (2003), Eskin and Ralston (2002) **3D**
  Uniqueness of Lamé coefficients λ, μ **from the DtN map if** $\lambda, \mu \in C^\infty(\overline{\Omega})$ **and** μ **is close to a constant**
- Nakamura and Uhlmann, 1993 **2D Uniqueness** $\lambda, \mu \in C^\infty(\overline{\Omega})$ **and** λ, μ **are close to a constant**
- Imanuvilov, Yamamoto, (2013) **2D**
  Global result for C^{10} **Lamé coefficients**.

Uniqueness

Case $\omega \neq 0$
Acoustic time harmonic waves

$$\nabla \cdot (\gamma \nabla u) + q\omega^2 u = 0$$

Nachman (1988): **Uniqueness of $\gamma \in C^2$ and $q \in L^\infty$ with DtoN maps at two different admissible frequencies**.
$a > 0$, $c > 0$

$$-\nabla \cdot (a \nabla u) + cu = 0,$$

Arridge-Lionheart (1998) **nonuniqueness**
Harrach (2012) **uniqueness piecewise constant diffusion and absorption**

Conditional Stability

Concerning stability: logarithmic (or worse) one is expected, (conductivity inverse problem: Mandache 2001).

Strategy:

Look for a-priori assumptions on the unknown parameters

- physically relevant
- give rise to a better type of stability

Regularization

- **Small volume conductivity inhomogeneities**: Friedman-Vogelius (1989), Vogelius-Fenjia -Moskow (1998), Ammari-B.-Francini (2006).....
- **Small volume elastic inhomogeneities**: Ammari-Kang-Nakamura-Tanuma (2002), B.-Bonnetier-Francini-Mazzucato (2012)
- **Finite parametrization of conductivities**

$$\gamma = \sum_{j=1}^{N} \gamma_j \chi_{D_j}, \quad \bigcup_{j=1}^{N} \overline{D}_j = \Omega$$

Alessandrini-Vessella (2005), B.-Francini (2011) (complex case), Gaburro-Sincich (2015) (anisotropic case)

$\Rightarrow$ **Lipschitz stability estimates**

Part III

Main results

Main assumptions

$$\mathbb{C}(x) = \sum_{m=1}^{N} (\lambda_m \delta_{ij}\delta_{kl} + \mu_m(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}))\chi_{D_m}(x)$$

$$\rho(x) = \sum_{m=1}^{N} \rho_m \chi_{D_m}(x)$$

$\{D_j\}_{j=1}^{N}$, *disjoint Lipschitz, partition of* Ω.

Main goal

1. Parameter estimation (determination of $\lambda_j, \mu_j, \rho_j, j = 1, \cdots, N$),
2. Interface identification (determination of $D_j, j = 1, \cdots, N$)

Part IV

Parameter estimation

Main assumptions on the partition $\{D_j\}_{1\leq j\leq N}$ and on Ω

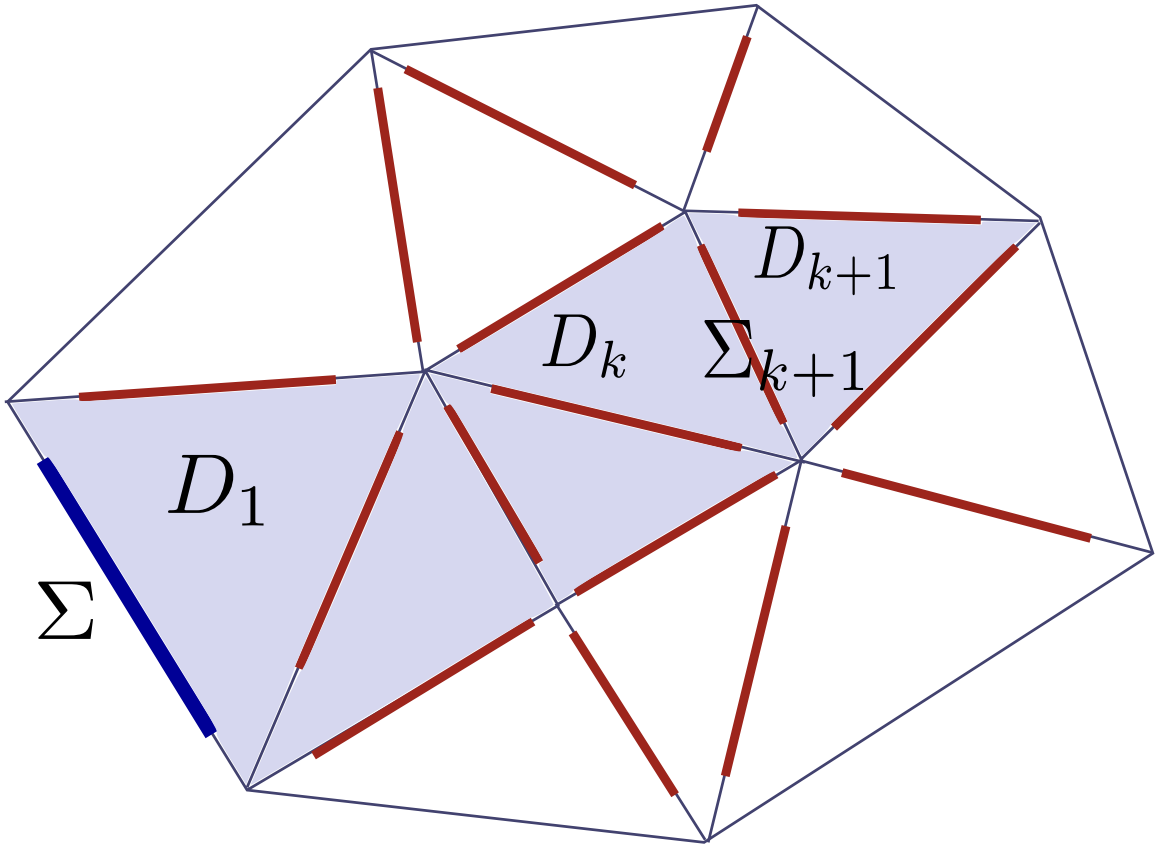
- **(H1)** Ω, $\{D_j\}_{1\leq j\leq N}$ Lipschitz domains with constants L, r_0
- **(H2)** $|\Omega| \leq A r_0^3$
- **(H3)** $\{D_j\}_{1\leq j\leq N}$ is a partition of Ω,
- **(H4)** $\partial D_1 \cap \partial\Omega := \Sigma$ is flat and $\forall j \in \{2,\dots,N\}$ there exists $j_1,\dots,j_M \in \{1,\dots,N\}$ such that

$$D_{j_1} = D_1, \quad D_{j_M} = D_j,$$

and

$$\Sigma_k \subset \partial D_{j_{k-1}} \cap \partial D_{j_k} \subset \Omega \ , \Sigma_k \text{ flat }, \quad \forall k = 2,\dots,M$$

Polyhedral partition



Part V

The main results

Local DN map

Local Dirichlet to Neumann map

$$H^{1/2}_{co}(\Sigma) = \left\{ \phi \in H^{1/2}(\partial\Omega) \ : \ \text{supp}\ \phi \subset \Sigma \subset \partial\Omega \right\}$$

$H^{-1/2}_{co}(\Sigma)$ topological dual of $H^{1/2}_{co}(\Sigma)$.
The local Dirichlet to Neumann is

$$\Lambda^{\Sigma}_{\mathbb{C},\rho} : \psi \in H^{1/2}_{co}(\Sigma) \to (\mathbb{C}\widehat{\nabla} u)n|_{\Sigma} \in H^{-1/2}_{co}(\Sigma).$$

Local DN map

- Identify $\Lambda^{\Sigma}_{\mathbb{C},\rho}$ with the bilinear form on $H^{1/2}_{co}(\Sigma) \times H^{1/2}_{co}(\Sigma)$ by

$$\widetilde{\Lambda}^{\Sigma}_{\mathbb{C},\rho}(\psi,\phi) := < \Lambda^{\Sigma}_{\mathbb{C},\rho}\psi, \phi > = \int_{\Omega} \mathbb{C}\widehat{\nabla} u : \widehat{\nabla} v - \omega^2 \rho u \cdot v$$

  $\forall \psi, \phi \in H^{1/2}_{co}(\Sigma)$ and where u is a solution to the BVP with datum ψ and v is any $H^1(\Omega)$ function s. t. $v = \phi$ on $\partial\Omega$

- Denote by

$$\| T \|_{\star} = \sup\{< T\psi, \phi > | \psi, \phi \in H^{1/2}_{co}(\Sigma), \|\psi\|_{H^{1/2}_{co}(\Sigma)} = \|\phi\|_{H^{1/2}_{co}(\Sigma)} = 1\}$$

  for every $T \in \mathcal{L}\left(H^{1/2}_{co}(\Sigma), H^{-1/2}_{co}(\Sigma)\right)$.

Main stability result

B., de Hoop, Francini, Vessella, Zhai (2017)

Theorem

Let $\omega^2 \in (0, \frac{\gamma_0 \lambda_1^0}{2}]$. Then, there exists a positive constant C depending on $L, A, N, \alpha_0, \beta_0, \gamma_0, \lambda_1^0$ such that, for any $\mathbb{C}^k$, $k = 1, 2$ and ρ^k we have

$$d_\Omega((\mathbb{C}^1, \rho^1), (\mathbb{C}^2, \rho^2)) \leq C \|\Lambda^\Sigma_{\mathbb{C}^1, \rho^1} - \Lambda^\Sigma_{\mathbb{C}^2, \rho^2}\|_\star.$$

where

$$d_\Omega((\mathbb{C}^1, \rho^1), (\mathbb{C}^2, \rho^2)) = \max\{\|\lambda^1 - \lambda^2\|_{L^\infty(\Omega)}, \|\mu^1 - \mu^2\|_{L^\infty(\Omega)}, \|\rho^1 - \rho^2\|_{L^\infty(\Omega)}\}$$

Static case ($\omega = 0$)
B. Francini, S. Vessella (2014)
Extension to case of $C^{1,\alpha}$ interfaces B. Francini, Morassi, Rosset, Vessella (2015)

Main ingredients of the proof

Proof: constructive and based on an iterative procedure

- QUANTITATIVE AND GLOBAL FORM OF THE INVERSE MAP THEOREM

- SINGULAR SOLUTIONS: Construction of Green's function with singularity close to Σ and study of asymptotic behaviour near Σ.

- QUANTITATIVE ESTIMATES OF UNIQUE CONTINUATION FOR SOLUTIONS OF ELLIPTIC SYSTEMS
  Unique continuation property: Let $v \in H^1(K)$ be a weak solution to

$$\operatorname{div}\left(\mathbb{C}\widehat{\nabla} v\right) + \omega^2 \rho v = 0 \quad \text{in} \quad K,$$

  which vanishes in an open subset $K_0 \subset K$ with $\mathbb{C}$ constant isotropic tensor. Then $v = 0$ in K.

Quantitative inverse function theorem

PROPOSITION - Bacchelli-Vessella 2006.

Let $\mathbf{K} \subset \mathcal{A} \subset \mathbb{R}^d$, $\mathbf{K}$ compact, $\mathcal{A}$ open,

$$\operatorname{dist}\left(\mathbf{K}, \mathbb{R}^d \setminus \mathcal{A}\right) \geq M_1, \text{ and } \mathbf{K} \subset B_{M_2}(0).$$

$$F : \mathcal{A} \to \mathcal{B}$$

$\mathcal{B}$ Banach space.

Quantitative inverse function theorem

1. F is Fréchet differentiable;
2. F' uniformly continuous, $\sigma_1(\cdot)$ modulus of continuity of F' ;
3. $F_{|\mathbf{K}}$ is injective;
4. $(F_{|\mathbf{K}})^{-1} : F(\mathbf{K}) \to \mathbf{K}$ uniformly continuous , with modulus of continuity $\sigma_2(\cdot)$
5. $$\min_{x\in\mathbf{K},|h|=1} \left\| F'(x)[h] \right\|_{\mathcal{B}} \geq q_0 > 0.$$

Then

$$\|x_1 - x_2\|_{\mathbb{R}^d} \leq C\|F(x_1) - F(x_2)\|_{\mathcal{B}} \quad \forall x_1, x_2 \in \mathbf{K},$$

where $C = \max\{\frac{2M_1}{\sigma_2^{-1}(\delta_1)}, \frac{2}{q_0}\}$, for $\delta_1 = \frac{1}{2}\min\{\delta_0, M_2\}$ with $\delta_0 = \sigma_1^{-1}(\frac{q_0}{2})$.

Reformulation of the problem

- $\underline{L} := (\lambda_1, \ldots, \lambda_N, \mu_1, \ldots, \mu_N, \rho_1, \ldots, \rho_N) \in \mathbb{R}^{3N}$;
- $$\mathbb{C}_{\underline{L}} = \sum_{j=1}^{N} \left(\lambda_j I_3 \otimes I_3 + 2\mu_j \mathbb{I}_{Sym}\right) \chi_{D_j}(x),$$
- $$\|\underline{L}\|_\infty = \max_{j=1,\ldots,N} \left\{\max\{|\lambda_j|, |\mu_j|, |\rho_j|\}\right\}.$$

Reformulation of the problem

Let $\omega^2 \in (0, \frac{\gamma_0 \lambda_1^0}{2}]$ and let

$$F : \mathbf{K} \subset \mathcal{A} \subset \mathbb{R}^{3N} \to \mathcal{B} := \mathcal{L}\left(H_{co}^{1/2}(\Sigma), H_{co}^{-1/2}(\Sigma)\right)$$
$$F(\underline{L}) = \Lambda_{\underline{L}}^{\Sigma}$$

If F satisfies the assumptions of the proposition then

Theorem 1'

$$\|\underline{L}^1 - \underline{L}^2\|_\infty \leq C\|F(\underline{L}^1) - F(\underline{L}^2)\|_\star \quad \forall \underline{L}^1, \underline{L}^2 \in \mathbf{K}$$

C depends on $L, A, N, \alpha_0, \beta_0, \gamma_0, \lambda_1^0$ only.

Differentiability of F and Lipschitz continuity of F'

F is Frechét differentiable in $\mathcal{A}$ and

$$< F'(\underline{L})[\underline{H}]\psi, \phi >= \int_\Omega \mathbb{H}\widehat{\nabla} u_{\underline{L}} : \widehat{\nabla} v_{\underline{L}} - h\omega^2 u_{\underline{L}} \cdot v_{\underline{L}} \, dx$$

where $\mathbb{H} = \mathbb{C}_{\underline{H}}$ and $h = \rho_{\underline{H}}$.

The proof is straightforward being a simple consequence of definition of F and of F'.

Injectivity of $F_{|\mathbf{K}}$ and uniform continuity of $(F_{|\mathbf{K}})^{-1}$

A first very rough stability estimates for $(F_{|\mathbf{K}})^{-1}$ can be derived

$$\sigma(t) = \begin{cases} |\log t|^{-\eta} & \text{for} \quad 0 < t < \frac{1}{e}, \\ t - \frac{1}{e} + 1 & \text{for} \quad t \geq \frac{1}{e}, \end{cases}$$

where η depend on A, L, α_0, β_0, N only

$$\|\underline{L}^1 - \underline{L}^2\|_\infty \leq C_* \sigma^N(\|F(\underline{L}^1) - F(\underline{L}^2)\|_\star) \quad , \forall \underline{L}^1, \underline{L}^2 \in \mathbf{K}$$

where

- $\sigma^N(\cdot) = \sigma \circ \cdots \circ \sigma$, ($N$ times)
- C_* depends on A, L, α_0, β_0, γ_0,λ_1^0, N only

Proof is based on construction of singular solutions and on estimates of unique continuation for solutions of elliptic systems

Injectivity of F' and lower bound of F'

Theorem

Let

$$q_0 := \min\{\|F'(\underline{L})[\underline{H}]\|_\star \mid \underline{L} \in \mathbf{K}, \underline{H} \in \mathbb{R}^{3N}, \|\underline{H}\|_\infty = 1\};$$

we have

$$(\sigma_1^N)^{-1}(1/C_\star) \leq q_0, \tag{1}$$

where $C_\star > 1$ depends on $A, L, \alpha_0, \beta_0, \gamma_0, \lambda_1^0$ and N only.

Follows with similar arguments as those used to prove continuity of $(F_{|\mathbf{K}})^{-1}$

Lipschitz stability in terms of the local Neumann to Dirichlet map

Local Neumann to Dirichlet map

$$H^{1/2}_{co}(\Sigma) = \left\{ \phi \in H^{1/2}(\partial\Omega) \ : \ \text{supp}\ \phi \subset \Sigma \subset \partial\Omega \right\}$$

$$H^{-1/2}(\Sigma) = \left\{ \psi \in H^{-1/2}(\partial\Omega) \ : < \psi, f >= 0, \ \forall f \in H^{1/2}_{co}(\Sigma) \right\}$$

$$\mathcal{N}^{\Sigma}_{\mathbb{C},\rho} : \psi \in H^{-1/2}(\Sigma) \to u|_{\Sigma} \in (H^{-1/2}(\Sigma))^* \subset H^{1/2}(\partial\Omega)$$

Lipschitz stability in terms of the Local NtoD map

Theorem

There exists a positive constant C depending on $L, A, N, \alpha_0, \beta_0, \gamma_0, \bar{\lambda}_1^0$ such that, for any $\mathbb{C}^k$, $k = 1, 2$ and ρ^k satisfying the stated assumptions and frequency $\omega^2 \in (0, \frac{\gamma_0 \bar{\lambda}_1^0}{2}]$

$$d_\Omega((\mathbb{C}^1, \rho^1), (\mathbb{C}^2, \rho^2)) \leq C \|\mathcal{N}^\Sigma_{\mathbb{C}^1, \rho^1} - \mathcal{N}^\Sigma_{\mathbb{C}^2, \rho^2}\|_\star$$

where $\bar{\lambda}_1^0$ is the first positive Neumann eigenvalue of the (known) operator $\mathbb{C}_0$ introduced previously.

Lipschitz stability in terms of the local NtoD map

Main Assumption

The elasticity tensor and density are known in a neighborhood of $\Sigma(D_1)$

Theorem

There exists a positive constant C depending on $L, A, N, \alpha_0, \beta_0, \gamma_0, \bar{\lambda}_1^0$ such that, for any $\mathbb{C}^k$, $k = 1, 2$ and ρ^k satisfying the stated assumptions and frequency $\omega^2 \in (0, \frac{\gamma_0 \bar{\lambda}_1^0}{2}]$

$$d_\Omega((\mathbb{C}^1, \rho^1), (\mathbb{C}^2, \rho^2)) \leq C \| \mathcal{N}^\Sigma_{\mathbb{C}^1, \rho^1} - \mathcal{N}^\Sigma_{\mathbb{C}^2, \rho^2} \|_{\mathcal{L}(L^2_{co}(\Sigma), L^2(\Sigma))}$$

where $L^2_{co}(\Sigma) = \{g \in L^2(\partial\Omega) : supp\, g \subset \Sigma\}$

Reconstruction algorithm

De Hoop, Qiu, Scherzer, 2012

Lipschitz stability estimates $\Rightarrow$ local convergence of iterative reconstruction methods

$m = (\mathbb{C}, \rho)$ and $m^\dagger = (\mathbb{C}^\dagger, \rho^\dagger)$ where $m^\dagger$ represents the parameter corresponding to the true model.

- $m = m^\dagger$ in $D_1 \Longrightarrow \mathcal{N}_m^\Sigma - \mathcal{N}_{m^\dagger}^\Sigma$ Hilbert Schmidt operator
- Consider for n sufficiently large the functional

$$J(m) = \frac{1}{2}\sum_{j=1}^{n} \|(\mathcal{N}_m^\Sigma - \mathcal{N}_{m^\dagger}^\Sigma)g_j\|^2_{L^2(\Sigma)} \approx \frac{1}{2}\|\mathcal{N}_m^\Sigma - \mathcal{N}_{m^\dagger}^\Sigma\|^2_{HS}$$

where $\{g_j\}_1^\infty$ is an orthonormal basis in $L^2_{co}(\Sigma)$.

Minimization of the Misfit Functional

Constrained Minimization

$$\arg\min_{m \in \mathbf{K}} J(m) = \frac{1}{2} \sum_{j=1}^{n} \int_{\Sigma} |u^j(m) - \bar{u}^j_{meas}|^2 \, ds$$

u^j weak solution to

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla} u^j) + \omega^2 \rho u^j &= 0 \text{ in } \Omega \subset \mathbb{R}^3 \\ (\mathbb{C}\widehat{\nabla} u^j)\nu &= g^j \text{ on } \partial\Omega \end{cases}$$

Computation of the gradient

To compute the gradient of the functional J we use Lagrangian approach

$$\mathcal{L}(m, u^1, \dots, u^n, v^1, \dots v^n) = J(m) + \sum_{i=1}^{n} F(v^i) - a(m, u^i, v^i)$$

where

$$F(v^i) = \int_\Sigma g^i \cdot v^i$$

and

$$a(m, u^i, v^i) = \int_\Omega \mathbb{C}\widehat{\nabla} u^i(x) : \widehat{\nabla} v^i(x) - \omega^2 \rho u^i(x) \cdot v^i(x)$$

Each v^i is Lagrange multiplier associated to a particular constraint i.e. the state problem with datum g^i.

Computation of the gradient of J

From the optimality conditions for $\mathcal{L}$ and the chain rule it is easy to see that

$$\nabla_m J(m) = \mathcal{L}_m(\tilde{u}^1(m), \dots, \tilde{u}^n(m), \tilde{v}^1(m), \dots \tilde{v}^n(m), m)$$

where $\tilde{u}^1(m), \dots, \tilde{u}^n(m)$ are the state solutions of the direct problem with datum $g^1, \dots, g^n$ and $\tilde{v}^1(m), \dots, \tilde{v}^n(m)$ are the adjoint state solutions satisfying

$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\tilde{v}^j) + \omega^2\rho\tilde{v}^j &= 0 \text{ in } \Omega \subset \mathbb{R}^3 \\ (\mathbb{C}\widehat{\nabla}\tilde{v}^j)\nu &= \chi_\Sigma(\tilde{u}^j - u^j_{meas}) \text{ on } \partial\Omega \end{cases}$$

Computation of the gradient of J

$$\nabla_{\mu_i} J(m) = \sum_{j=1}^{n} \int_{D_i} \widehat{\nabla} \tilde{u}^j(x) : \widehat{\nabla} \tilde{v}^j(x)$$

$$\nabla_{\lambda_i} J(m) = \sum_{j=1}^{n} \int_{D_i} \nabla \cdot \tilde{u}^j(x) \nabla \cdot \tilde{v}^j(x)$$

$$\nabla_{\rho_i} J(m) = \sum_{j=1}^{n} \int_{D_i} \tilde{u}^j(x) \cdot \tilde{v}^j(x)$$

Algorithm: Multi-level multifrequency scheme

- Fix a (small) frequency and a coarse model representation
- Start from an initial guess and iterate a gradient method
- Increase the frequency and fix a finer model representation and again iterate a gradient method starting from the solution of the previous step as initial guess

Numerical results: Shallow bodies

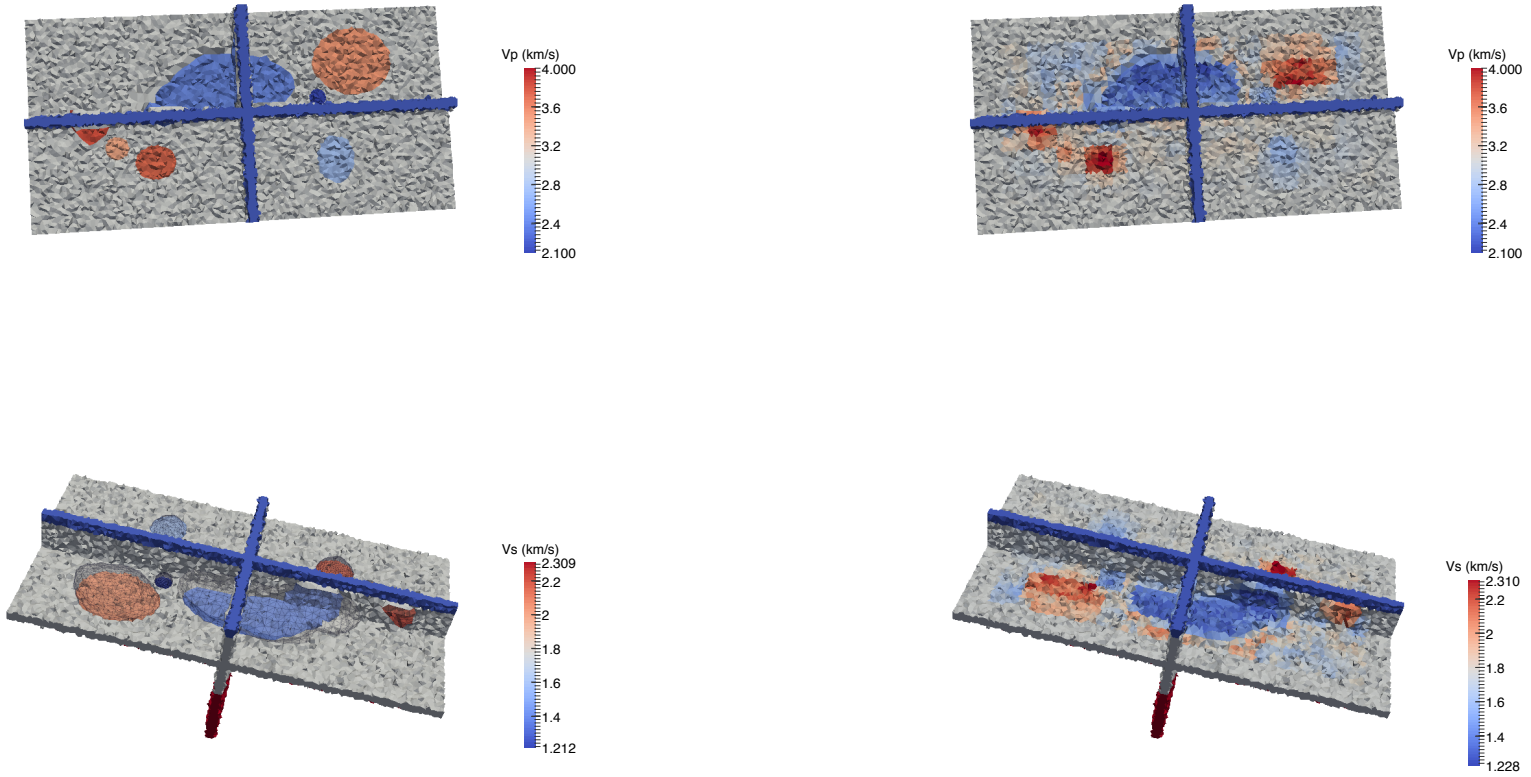


Figure: On the left: true model of the compressional wavespeed V_p and of the shear wavespeed V_s . On the right final reconstruction of V_p, V_s

Numerical results: Deep bodies

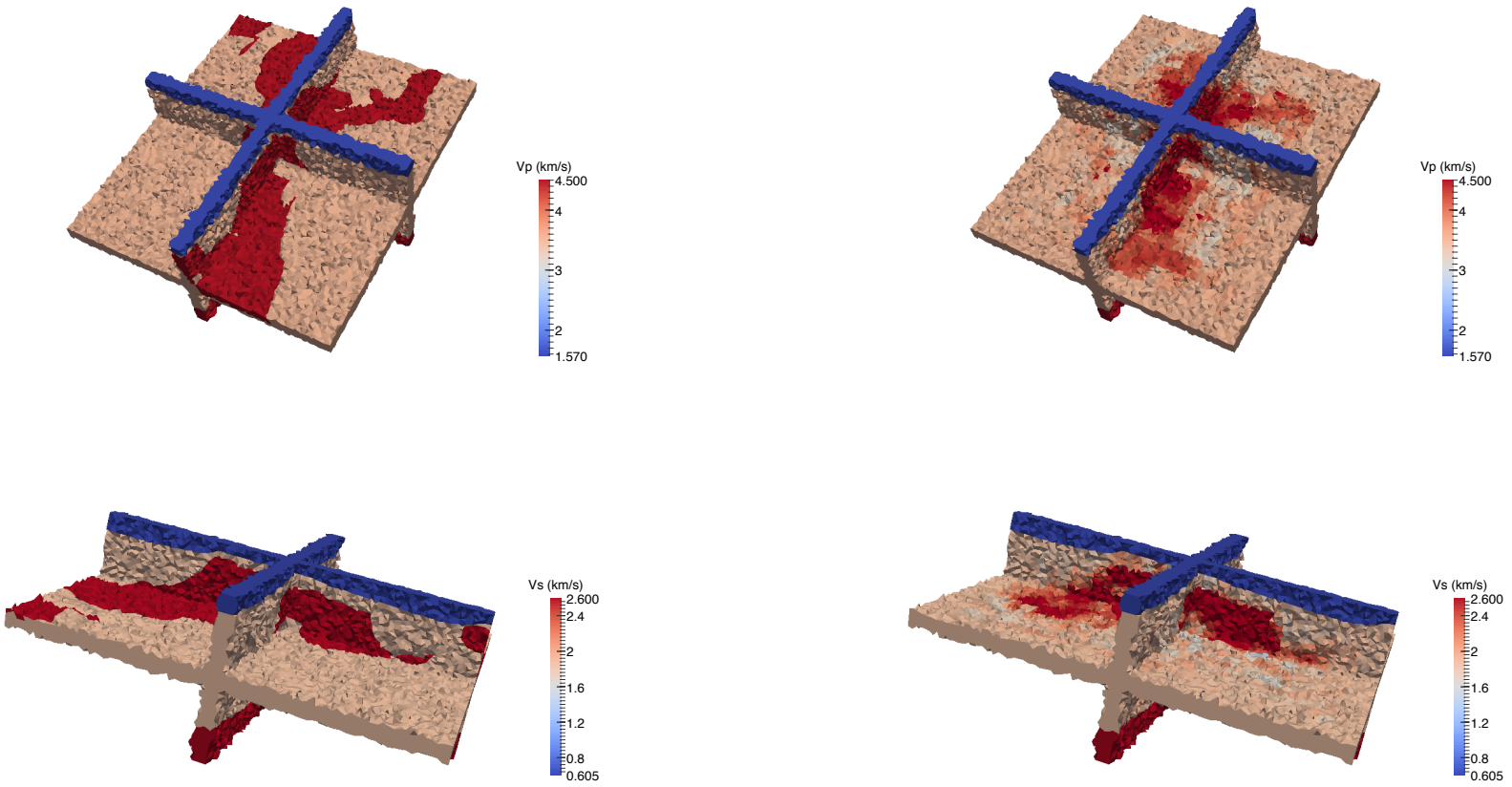


Figure: On the left: true model of V_p and V_s. On the right final reconstruction of V_p, V_s

Part VI

Interface identification

Interface identification

$$\mathbb{C}(x) = \sum_{m=1}^{N} (\lambda_m \delta_{ij}\delta_{kl} + \mu_m(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}))\chi_{D_m}(x)$$

$$\rho(x) = \sum_{m=1}^{N} \rho_m \chi_{D_m}(x)$$

"known" parameters λ_j, μ_j, ρ_j
unknown domains D_j
Highly nonlinear problem : Identification of a single elastic smooth interface (log-log stability)
Alessandrini-Di Cristo-Morassi-Rosset (2014)

Acoustic time-harmonic wave equation

$$\Delta u + \omega^2 q u = 0 \quad \text{in } \Omega \quad \text{for } q = \sum_{j=1}^{N} q_j \chi_{D_j}$$

Theorem

Let

$$q^0 = \sum_{j=1}^{N} q_j^0 \chi_{D_j^0}, \quad q^1 = \sum_{j=1}^{M} q_j^1 \chi_{D_j^1},$$

with $\{D_j^k\}$ *regular partitions of* **tetrahedra**, q_j^k *in a given set of finite, "distinguished" positive values, and* ω *small enough, there exist* ϵ_0 *and* C_0, *such that if* $\|\Lambda_{q^0} - \Lambda_{q^1}\| \leq \epsilon_0$ *then*

$$N = M, \quad q_j^0 = q_j^1 \quad \text{and} \quad d_{\mathcal{H}}(D_j^0, D_j^1) \leq C_0 \|\Lambda_{q^0} - \Lambda_{q^1}\|$$

B.- Francini-De Hoop-Vessella (2015)

First step towards elasticity case: conductivity equation in 2D

Consider

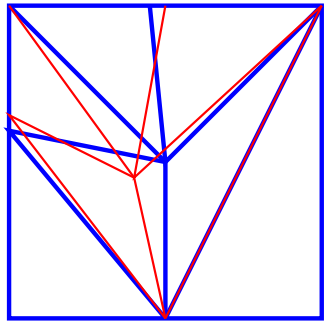
$$F : \mathcal{V} \subset \mathbb{R}^{dN} \to \mathcal{L}(H^{-1/2}(\Sigma), (H^{-1/2}(\Sigma))^*)$$

where $\mathcal{V}$ is the set of vertices of the polygonal partition.

Crucial step: Differentiability properties of the map F with respect to vertices and determination of Frechet derivative.

Difficulty: Gradients of solutions may blow up at vertices.

$$\gamma^t = \sum_{j=1}^{N} q_j^0 \chi_{D_j^t}$$



$P_{j,i}^0$ (vertex of D_j^0) $\Rightarrow P_{j,i}^0 + tV_{i,j}$ (vertex of D_j^t)

First step towards elasticity case: conductivity equation

Let u_0 and v_0 be solutions to

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla u_0) &= 0 \text{ in } \Omega, \\ u_0 &= f \text{ on } \partial\Omega. \end{cases} \qquad \begin{cases} \operatorname{div}(\gamma_0 \nabla v_0) &= 0 \text{ in } \Omega, \\ v_0 &= g \text{ on } \partial\Omega. \end{cases}$$

Theorem

$$\frac{d}{dt} < \Lambda_{\gamma_t} f, g >_{t=0} = (k-1) \int_{\partial P^0} (M_0 \nabla u_0^e \cdot \nabla v_0^e)(\Phi_0^{\vec{V}} \cdot n_0)\, d\sigma.$$

with $M_0 = \tau_0 \otimes \tau_0 + \frac{1}{k} n_0 \otimes n_0$ *where* τ_0 *and* n_0 *are the tangent and outer normal directions on* ∂P^0 *and* $\Phi_0^{\vec{V}}$ *is the affine function such that*

$$\Phi_0^{\vec{V}}(P_j^0) = V_j, \text{ for } j = 1, 2, 3.$$

B.-Francini-Vessella (2017)

Extends to Voronoi partitions. Extension to arbitrary partitions not known.

Concluding remarks

1. Prove differentiability of the Dirichlet to Neumann map (Neuman to Dirichlet map) with respect to vertices and find the derivative in the elastic case (2D and 3D)
2. Use differentiability properties to prove Lipschitz stability analogous to the acoustic time harmonic case.
3. Compute shape derivative of misfit functional to implement a shape optimization based algorithm to recover subsurface rough elastic interfaces
   conductivity problem: B.-Micheletti-Perotto-Santacesaria (2017)

Thank you for your attention!

Biphase fundamental solution

Consider the isotropic tensor

$$\mathbb{C} = \mathbb{C}^{+}\chi_{\mathbb{R}^3_+} + \mathbb{C}^{-}\chi_{\mathbb{R}^3_-}$$

where

$$\mathbb{C}^{+} = \lambda I_3 \otimes I_3 + 2\mu \mathbb{I}_{sym}, \quad \mathbb{C}^{-} = \lambda' I_3 \otimes I_3 + 2\mu' \mathbb{I}_{sym}.$$

Explicit biphase fundamental solution

$$\mathrm{div}\left(\mathbb{C}\widehat{\nabla}\Gamma(\cdot, y)\right) = \delta_y I_3$$

Rongved (1955).

Construction of the Green's function

- Extend Ω to a new domain $\Omega_0 := \Omega \cup D_0$ such that $\partial\Omega_0$ is of Lipschitz class.
- Extend the unknown tensor $\mathbb{C}$ and density ρ to Ω_0 (still denote by $\mathbb{C}$ the extension) such that

$$\mathbb{C}_{|D_0} A = 2\hat{A} \quad \text{, for every matrix } A$$

$$\rho_{|D_0} = 1$$

- $$\mathbb{C} = \mathbb{C}_0 \chi_{D_0}(x) + \sum_{j=1}^{N} \mathbb{C}_j \chi_{D_j}(x)$$

$$\rho = \rho_0 \chi_{D_0}(x) + \sum_{j=1}^{N} \rho_j \chi_{D_j}(x)$$

Green's function

Theorem

Let $\mathcal{D} = \cup_{j=0}^{N} \partial D_j \setminus \cup_{k=0}^{N} \Sigma_k$. $\forall y \in \Omega_0 \setminus \mathcal{D}$, *there exists* $G(\cdot, y)$, *continuous in* $\Omega_0 \setminus \{y\}$ *s. t.*

$$\int_{\Omega_0} \mathbb{C} \widehat{\nabla} G(\cdot, y) : \widehat{\nabla} \phi - \omega^2 \rho G(\cdot, y) \cdot \phi = \phi(y), \text{ for every } \phi \in C_0^\infty(\Omega_0).$$

$$G(\cdot, y) = 0 \quad \text{on } \partial\Omega_0.$$

$$G(x, y) = G(y, x)^T \quad \text{for every } x, y \in \Omega_0 \setminus \mathcal{D}.$$

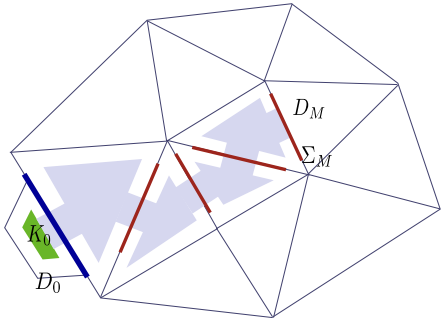
$$\|G(\cdot, y) - \Gamma(\cdot, y)\|_{H^1(\Omega_0)} \leq C \quad \text{if } \operatorname{dist}(y, \mathcal{D} \cup \partial\Omega_0)) \geq r_0/C$$

$$\|G(\cdot, y)\|_{H^1(\Omega_0 \setminus B_r(y))} \leq C r^{-1/2}, \quad \text{if } \operatorname{dist}(y, \mathcal{D} \cup \partial\Omega_0)) \geq r_0/C$$

Estimates of unique continuation

- **Three Spheres Inequality**: Alessandrini- Morassi, (2001)
- **Estimates for the Cauchy problem**: Morassi-Rosset, (2004)
- **Regularity C^∞ at flat interfaces**: Chipot-Kinderlehrer-Vergara Caffarelli (1986), Li-Nirenberg, (2003)

Estimates of unique continuation



$$\operatorname{div}(\mathbb{C}\nabla v) + \omega^2 \rho v = 0 \quad \text{in} \quad \mathcal{K},$$

$$\|v\|_{L^\infty(K_0)} \leq \epsilon_0,$$

$$|v(x)| \leq (\epsilon_0 + E_0)\left(\frac{\operatorname{dist}(x, \Sigma_M)}{r_0}\right)^{-\frac{1}{2}} \quad \text{for every} \quad x \in \mathcal{K}_h.$$

Then

$$|v(\tilde{x}_r)| \leq C\left(\frac{r_0}{r}\right)^2 (E_0 + \epsilon_0)^{1-\theta^{mM}\tau_r} \epsilon_0^{\theta^{mM}\tau_r}$$

$\tilde{x}_r = P_M - 2r\nu(P_M)$, $P_M \in \Sigma_M$, $r \in (0, r_0)$ and $m, \tau_r, \theta \in (0,1)$ and C depend on $r_0, L, A, , \alpha_0, \beta_0, N$.

A sketch of the proof of uniform continuity of $(F_{|\mathbf{K}})^{-1}$

Set $\mathbb{C} := \mathbb{C}_{\underline{L}^1}$ and $\bar{\mathbb{C}} := \mathbb{C}_{\underline{L}^2}$, $\rho := \rho_{\underline{L}^1}$ and $\bar{\rho} := \rho_{\underline{L}^2}$

$$\varepsilon := \|\Lambda^{\Sigma}_{\mathbb{C},\rho} - \Lambda^{\Sigma}_{\bar{\mathbb{C}},\bar{\rho}}\|_\star$$

Consider the Green's functions in $\Omega_0 = \Omega \cup D_0$ corresponding to $(\mathbb{C}, \rho)$ and $(\bar{\mathbb{C}}, \bar{\rho})$ for $y, z \in K_0$ fixed. Then

$$\int_{\Omega_0} (\mathbb{C} - \bar{\mathbb{C}})\widehat{\nabla} G(\cdot, y) : \widehat{\nabla} \bar{G}(\cdot, z) - \omega^2(\rho - \bar{\rho}) G(x, y) \cdot \bar{G}(x, z) =$$
$$< (\Lambda_{\mathbb{C},\rho} - \Lambda_{\bar{\mathbb{C}},\bar{\rho}})\bar{G}(\cdot, z), G(\cdot, y) >$$

Hence, since $dist(K_0, \partial D_0) \geq r_0/C$,

$$\left| \int_{\Omega_0} (\mathbb{C} - \bar{\mathbb{C}})\widehat{\nabla} G(\cdot, y) : \widehat{\nabla} \bar{G}(\cdot, z) - \omega^2(\rho - \bar{\rho}) G(x, y) \cdot \bar{G}(x, z) \right| \leq C\varepsilon$$

Sketch of the proof

Define, for $y, z \in K_0$

$$\mathcal{S}_0(y,z) := \int_{\Omega_0 \setminus D_0} (\mathbb{C} - \bar{\mathbb{C}}) \widehat{\nabla} G(\cdot, y) : \widehat{\nabla} \bar{G}(\cdot, z) - \omega^2 (\rho - \bar{\rho}) G(x,y) \cdot \bar{G}(x,z),$$

$\mathcal{S}_0(\cdot, z), \mathcal{S}_0(y, \cdot) \in H^1_{loc}(D_0)$ $\forall y, z \in D_0$

$$\mathrm{div}(\mathbb{C} \widehat{\nabla}_y \mathcal{S}_0^{(\cdot,q)}(\cdot, z)) + \omega^2 \rho \mathcal{S}_0^{(\cdot,q)}(\cdot, z)) = 0 \quad \text{in } D_0,$$

$$\mathrm{div}(\bar{\mathbb{C}} \widehat{\nabla}_z \mathcal{S}_0^{(p,\cdot)}(y, \cdot)) + \omega^2 \rho \mathcal{S}_0^{(p,\cdot)}(y, \cdot)) = 0 \quad \text{in } D_0.$$

$$|S_0(y,z)| \leq C\epsilon \quad \forall y, z \in K_0 \times K_0,$$

and

$$|\mathcal{S}_0(y,z)| \leq C(dist(y, \Sigma) dist(z, \Sigma))^{-1/2} \quad , \; y, z \in D_0,$$

Sketch of the proof

By smallness propagation estimates and regularity estimates of the gradient we have

$$|\mathcal{S}_0(y_r, z_r)| \leq C r^{-9/2} \varepsilon^{\tau_r}, |\partial_{y_1} \partial_{z_1} \mathcal{S}_0(y_r, z_r)| \leq C r^{-15/2} \varepsilon^{\tau_r'}$$

where y_r, z_r are points at distance r from Σ and y_1, z_1 are directions lying in Σ. Putting together the estimates of unique continuation with the asymptotic behaviour of the Green functions close to the interface we get

$$|\mu_1 - \bar{\mu}_1|, |\lambda_1 - \bar{\lambda}_1|, |\rho_1 - \bar{\rho}_1| \leq \omega_1(\varepsilon)$$

Sketch of the proof

Consider

$$\mathcal{S}_1(y,z) := \int_{\Omega_0 \setminus D_0 \cup D_1} (\mathbb{C} - \bar{\mathbb{C}}) \widehat{\nabla} G(\cdot, y) : \widehat{\nabla} \bar{G}(\cdot, z) - \omega^2 (\rho - \bar{\rho}) G(x,y) \cdot \bar{G}(x,z),$$

Then, for $y, z \in K_0$

$$|\mathcal{S}_1(y,z)| \leq C(\varepsilon + \omega_1(\varepsilon)),$$

$$|\mathcal{S}_1(y,z)| \leq C(dist(y,\Sigma) dist(z,\Sigma))^{-1/2} \quad , \; y, z \in D_0 \cup D_1,$$

So, similarly, using the properties of $\mathcal{S}_1$ we can show

$$|\mu_2 - \bar{\mu}_2|, |\lambda_2 - \bar{\lambda}_2|, |\rho_2 - \bar{\rho}_2| \leq \omega_2(\varepsilon)$$

and iterating the argument we get

$$|\mu_{k+1} - \bar{\mu}_{k+1}|, |\lambda_{k+1} - \bar{\lambda}_{k+1}|, |\rho_{k+1} - \bar{\rho}_{k+1}| \leq \omega_{k+1}(\varepsilon)$$

for all $k = 0, \ldots, M$